Matching with Transfers under Distributional Constraints

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Introduction and Motivation

- Many two-sided matching markets with **transferable utilities**, e.g., labor or rental housing markets, are subject to **distributional constraints**
- Prior work on matching under constraints has mainly focused on the non-transferable utility setting
- We study conditions on the constraint structure under which **equilibria exist** in the transferable utility setting

Definitions

An **arrangement** is specified by an assignment $\boldsymbol{X}$ and salaries $\{s_f\}_{f \in F}$, where F is the set of firms

Key Properties

1. Feasibility: An arrangement is feasible if the assignment $\boldsymbol{X}$ satisfies the distributional constraints

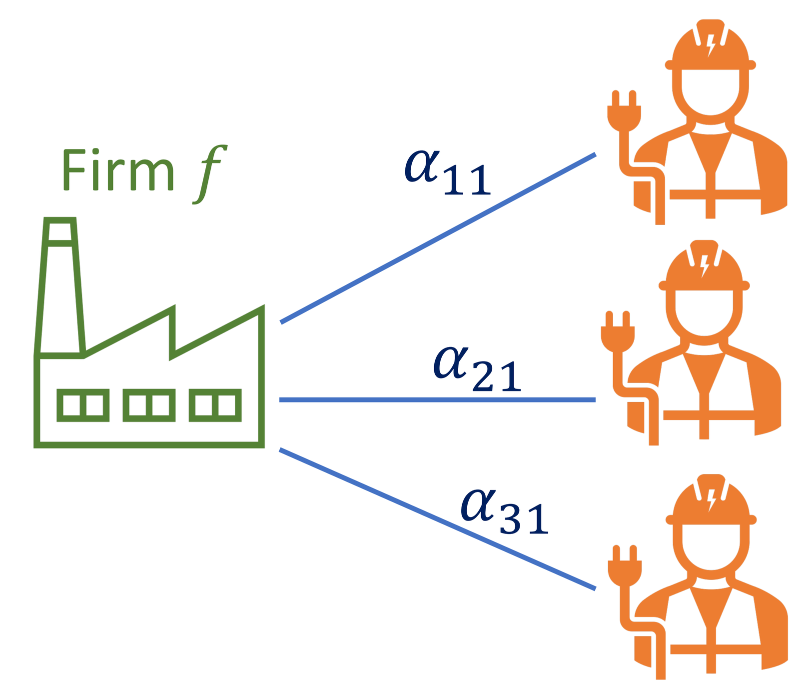
2. Stability: No firm and *feasible* group of workers can form a coalition, and all become better off by deviating

3. Efficiency: An assignment is *efficient* if it maximizes the total match value among all *feasible* assignments

Theorem (Efficiency of Stable Arrangements): *If a feasible arrangement is stable, then the corresponding assignment is efficient.*

Model

Total Match Value



a_{wf}: Value of firm f for worker w

c_{wf}: Value of worker w for firm f

$\alpha_{wf} = a_{wf} + c_{wf}$: Total Match Value of worker-firm pair w-f

Payoffs

Worker Payoff on matching with f:

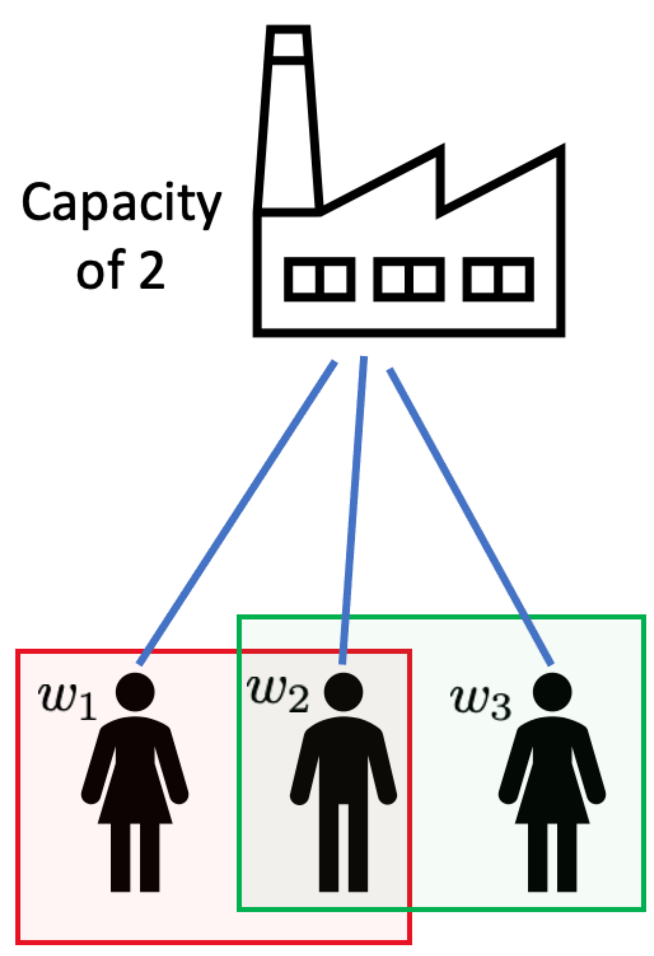
$$u_w = a_{wf} + s_{wf}$$

Firm Payoff on matching with feasible set of workers $D \in T_f$:

$$v_f = \sum_{w \in D} (c_{wf} - s_{wf})$$

Stable Arrangements for One Firm Setting

Example where Substitutes Condition is Violated



Constraints

$x_{w_1,f_1} + x_{w_2,f_1} \leq 1$

$x_{w_2,f_1} + x_{w_3,f_1} \leq 1$

Value of Worker for Firm	Value of Firm for Worker
$a_{11} = -0.5$	$c_{11} = 1.5$
$a_{21} = -0.5$	$c_{21} = 2.5$
$a_{31} = -0.5$	$c_{31} = 1.5$

Salary Profile 1	Salary Profile 2
$s_{11} = 0.5$	$s'_{11} = 1.1$
$s_{21} = 1$	$s'_{21} = 1$
$s_{31} = 0.5$	$s'_{31} = 0.5$

Increasing the salary of worker one decreases firm's demand for worker three

Substitutes condition: If the wage for one worker is increased, the demand for all other workers should weakly increase

Theorem (Existence of Stable Arrangement for One Firm): *In a one firm setting, stable arrangements exist irrespective of the nature of the constraint structure or agent's preferences.*

Takeaway: Stable arrangements exist even when substitutes condition is violated

Future Work

1. Extending linear programming approach to a broader range of firm preferences
2. Extending results to setting when preferences are not quasi-linear
3. Developing mechanisms that are incentive compatible in addition to preserving stability

Stable Arrangements in Multiple Firm Setting

Proposition: *Under general constraints,* ***stable arrangements may not exist****, even if firms have linear preferences*

$$\max_{\mathbf{X} \in \mathbb{R}^{|W| \times |F|}} \quad U(\mathbf{X}) = \sum_{w \in W} \sum_{f \in F} \alpha_{w,f} x_{w,f}, \quad \text{Total Match Value}$$

s.t.

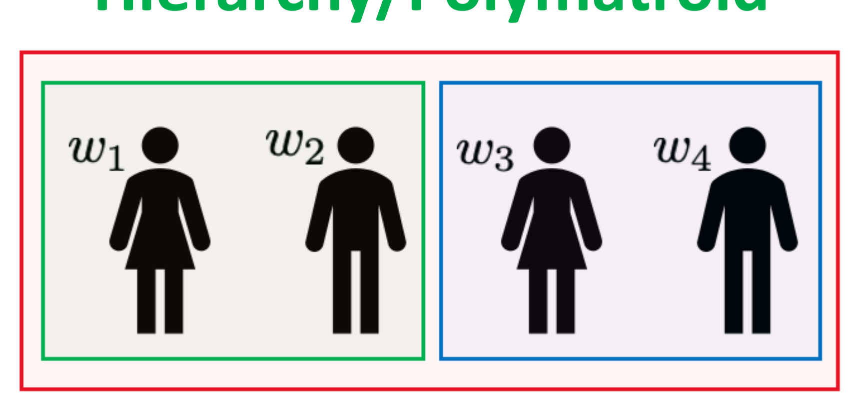
$$\sum_{f \in F} x_{w,f} \leq 1, \quad \forall w \in W, \quad \text{Worker Allocation Constraints}$$

$$x_{w,f} \geq 0, \quad \forall w \in W, f \in F,$$

$$\sum_{w \in D} x_{w,f} \leq \bar{\lambda}_D, \quad \forall D \in \mathcal{H}_f, f \in F. \quad \text{Distributional Constraints}$$

Theorem (Linear Programming and Stable Arrangements): *If the distributional constraints form a polymatroid (or hierarchy), then there exists a stable arrangement, and it can be computed in polynomial time using linear programming.*

Hierarchy/Polymatroid



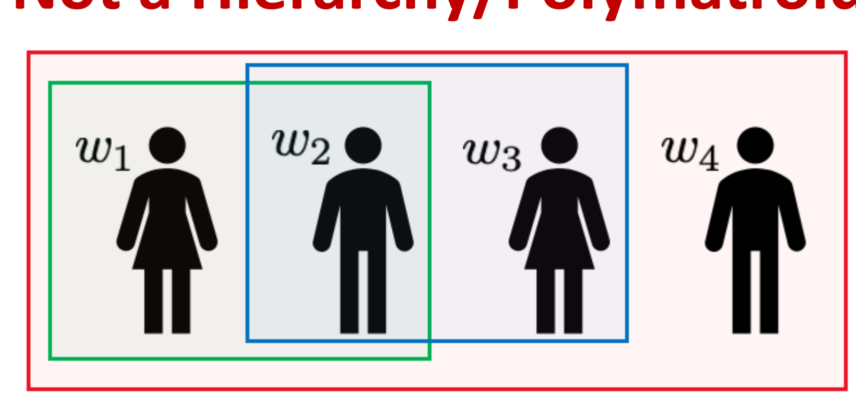
Constraints

$x_{w_1,f_1} + x_{w_2,f_1} + x_{w_3,f_1} + x_{w_4,f_1} \leq 2$

$x_{w_1,f_1} + x_{w_2,f_1} \leq 1$

$x_{w_3,f_1} + x_{w_4,f_1} \leq 1$

Not a Hierarchy/Polymatroid



Constraints

$x_{w_1,f_1} + x_{w_2,f_1} + x_{w_3,f_1} + x_{w_4,f_1} \leq 2$

$x_{w_1,f_1} + x_{w_2,f_1} \leq 1$

$x_{w_2,f_1} + x_{w_3,f_1} \leq 1$

Hierarchy: For all sets of workers $S, S' \in H_f$, either $S \subseteq S'$, $S' \subseteq S$ or $S \cap S' = \emptyset$

Linear Programming approach can be extended to **lower bound quotas** under a slight modification to the stability notion